Potential Flow

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Overview

- Refs.: chap. 4 of Acheson, chap, 10 of Çengel, Faber.
- For irrotational flow, $\nabla \times \vec{V} = 0$, which implies that $\vec{V} = \pm \nabla \phi$.
- ϕ is a scalar field called the potential flow function.

- If the fluid is incompressible, then the continuity equation implies that $\nabla \cdot \vec{V} = 0$.
- In this case the $\underbrace{\text{potential flow}}_{\text{escoamento potencial}}$ function satisfies the Laplace equation

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0}$$

- We can obtain many velocity fields using the techniques used to solve Laplace's equation.

Velocity field

Given the flow potential, the velocity field is obtained from its gradient:

Cartesian coordinates,

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

and in cylindrical coordinates,

$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad u_z = \frac{\partial \phi}{\partial z}$$

Cartesian Coordinates (x, y, z)

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \nabla\phi$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Cylindrical Coordinates (r, θ, z)

$$r^2 = x^2 + y^2, \ \theta = \tan^{-1}\left(y/x\right)$$

$$\vec{V} = u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_z\hat{e}_z = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z = \nabla\phi$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

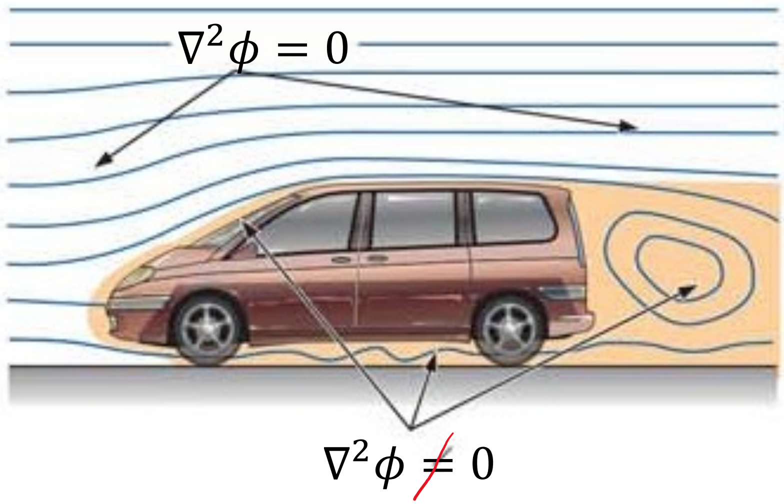
Spherical Coordinates (r, θ, φ)

$$r^2 = x^2 + y^2 + z^2, \ \theta = \cos^{-1}\left(x/r\right), \text{ or } x = r\cos\theta, \varphi = \tan^{-1}\left(z/y\right)$$

$$\vec{V} = u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_\varphi\hat{e}_\varphi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi = \nabla\phi$$

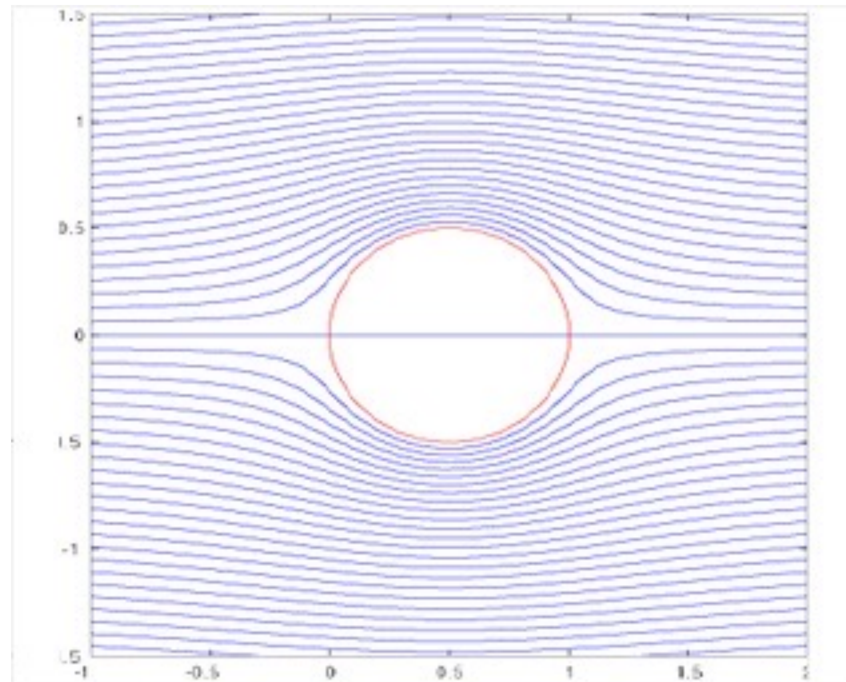
$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

Example (schematic)

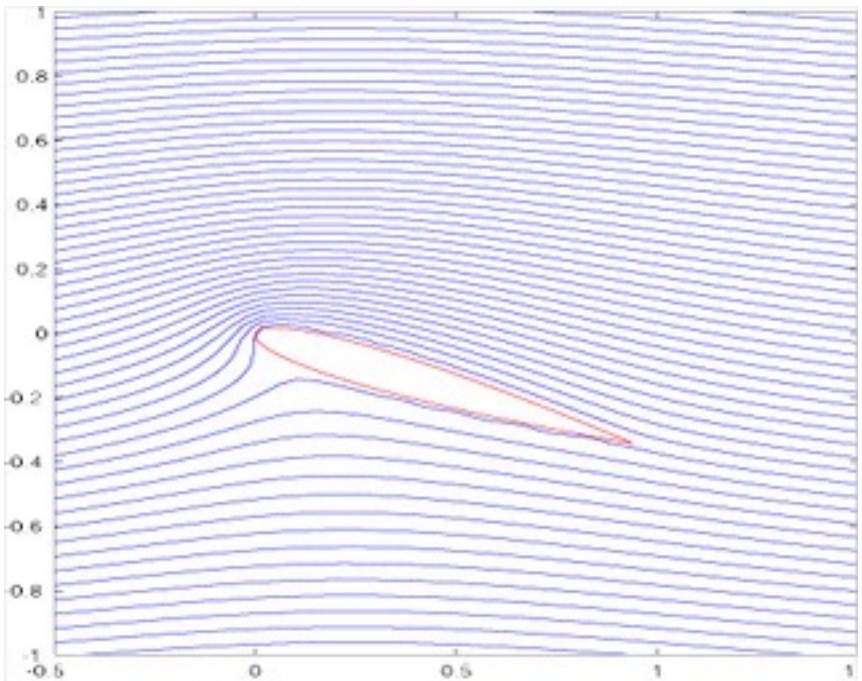


Examples (solutions of Laplace's equation)

Cylinder in free stream



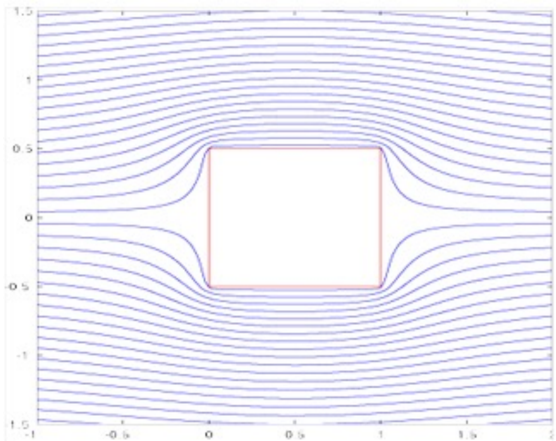
Airfoil in free stream

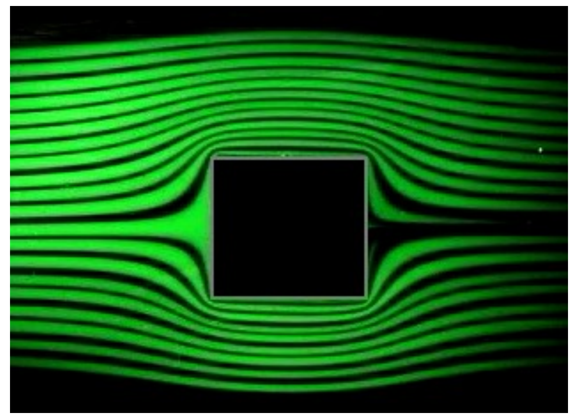


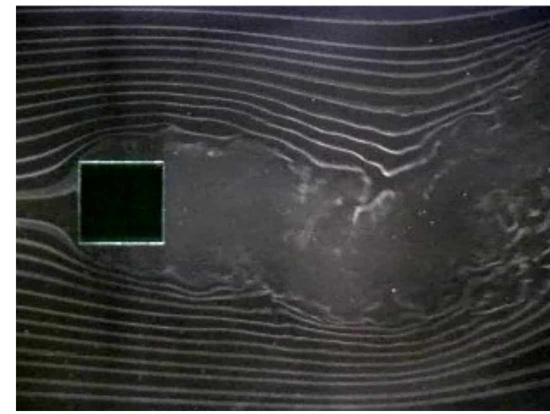
Examples

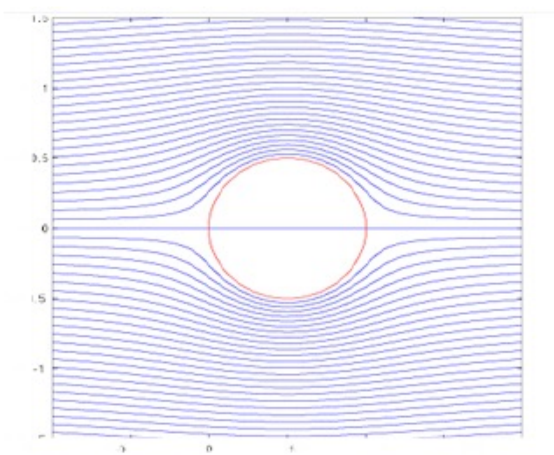
$Re < 1$

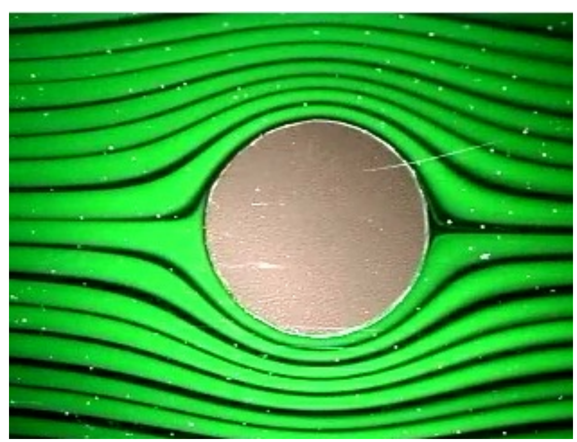
Re=10000

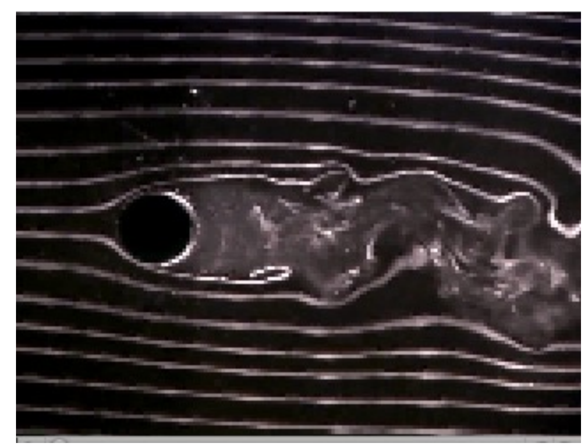




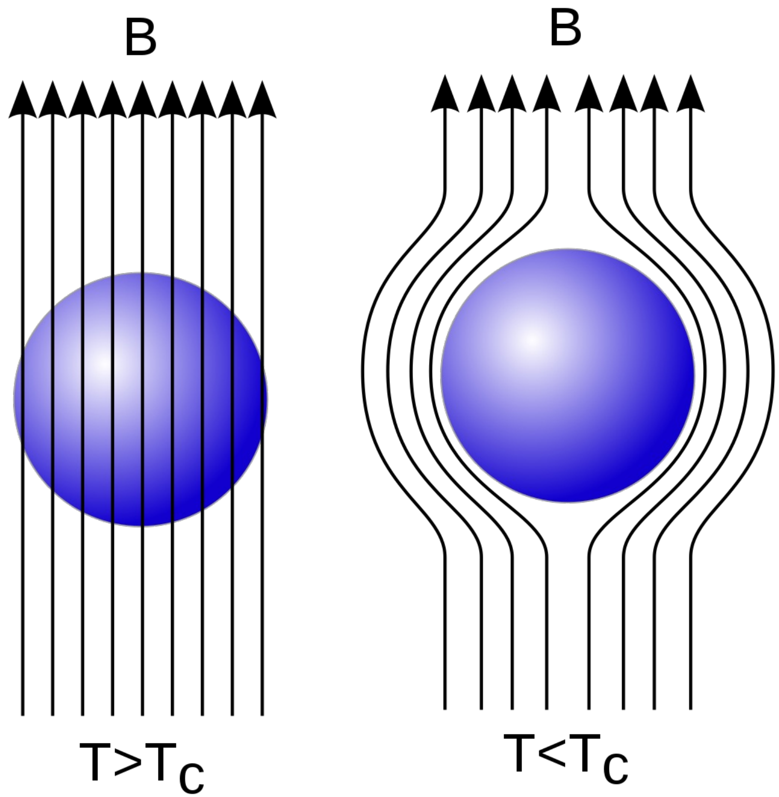








Superconductor



https://en.wikipedia.org/wiki/Meissner_effect

Ex.:

$$u = \alpha x, \qquad v = -\alpha y, \qquad w = 0$$

Back to Laplace's equation

For irrotational regions of flow: $$\nabla^2\phi = 0$$

In cartesian coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

In cylindrical coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Spherical and mixed coordinates may also be useful.

- The beauty of this is that we have combined three unknown velocity components (e.g., u, v, and w) into one unknown scalar field ϕ, eliminating two of the equations required for a solution.
- Once we obtain a solution, we can calculate all three components of the velocity field.

- The Laplace equation is well known since it shows up in several fields of physics, applied mathematics, and engineering. Various solution techniques, both analytical and numerical, are available in the literature.
- Solutions of the Laplace equation are dominated by the geometry (i.e., boundary conditions).

- The solution is valid for any incompressible fluid, regardless of its density or its viscosity, in regions of the flow in which the irrotational approximation is appropriate

Pressure

Of course we still need a dynamical equation to calculate the pressure field. This will be given by the Euler equation.

If gravity is the only body force, then

For irrotational regions of flow: $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g}$$

Or in its integrated form, the Bernoulli equation

Steady, incompressible flow: $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since the flow is irrotational, we can apply Bernoulli to ANY two points in the flow domain.

Stream function

- For irrotational flows in 2D, the stream function obeys the Laplace equation:

$$\nabla^2 \psi = 0.$$

- In potential 2D flow, both the flow potential and the stream function are solutions of the Laplace equation.

- Lines of constant flow potential are perpendicular to the streamlines (check).

- In axisymmetric flows the stream function obeys a linear equation but that is no longer Laplace's equation.

Stream function

For incompressible 2D flows:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Important property: ψ is constant along a streamline.

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + (\boldsymbol{u} \cdot \nabla)\psi = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0 \Rightarrow \frac{D\psi}{Dt} = 0$$

$\Rightarrow \psi = cte$ em uma linha de corrente

Generic coordinate system (only in 2D)

$$\boldsymbol{u} = \nabla \wedge (\psi \boldsymbol{k})$$

Complex potential

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

The complex potential is also a solution of the Laplace equation

$$w = \phi + \mathrm{i}\psi \quad = r e^{i\theta}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

19/10/2023

Resumo – Função de corrente (ψ)

– Esc. em 2D

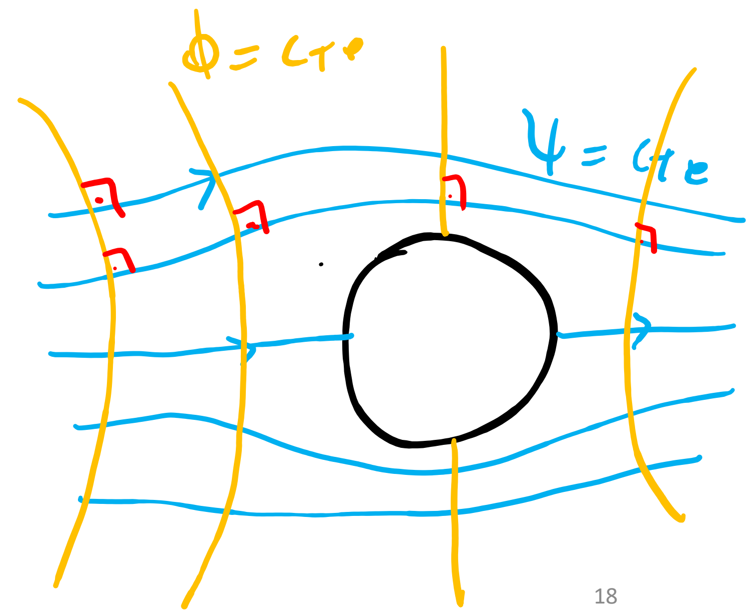
– Esc. potenciais $\begin{cases} \nabla \times \vec{u} = 0 \Rightarrow \nabla^2 \psi = 0 \\ \nu = 0 \\ \nabla \cdot \vec{u} = 0 \leftarrow \end{cases}$

– $\psi = cte$ ao longo de uma linha de corrente

C. Cartesianas

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

Em geral: $\vec{u} = \nabla \times (\psi \hat{k})$



Teorema da circulação de Kelvin

– Se $\vec{\omega} = \vec{0}$ em $t = 0$, $\vec{\omega} = 0$ em $t \neq t_0$

$$K = \oint_C \vec{u} \cdot d\vec{l} = \int_S \vec{\omega} \cdot d\vec{S}$$

$$\frac{DK}{Dt} = \underbrace{\oint_C \frac{D\vec{u}}{Dt} \cdot d\vec{l}}_{I_1} + \underbrace{\oint_C \vec{u} \cdot \frac{D\,d\vec{l}}{Dt}}_{I_2}$$

$$I_1 = \oint_c \frac{D\vec{u}}{Dt} \cdot d\vec{l} = -\oint_c \nabla\left(\frac{p}{\rho} + gz\right) \cdot d\vec{l}$$

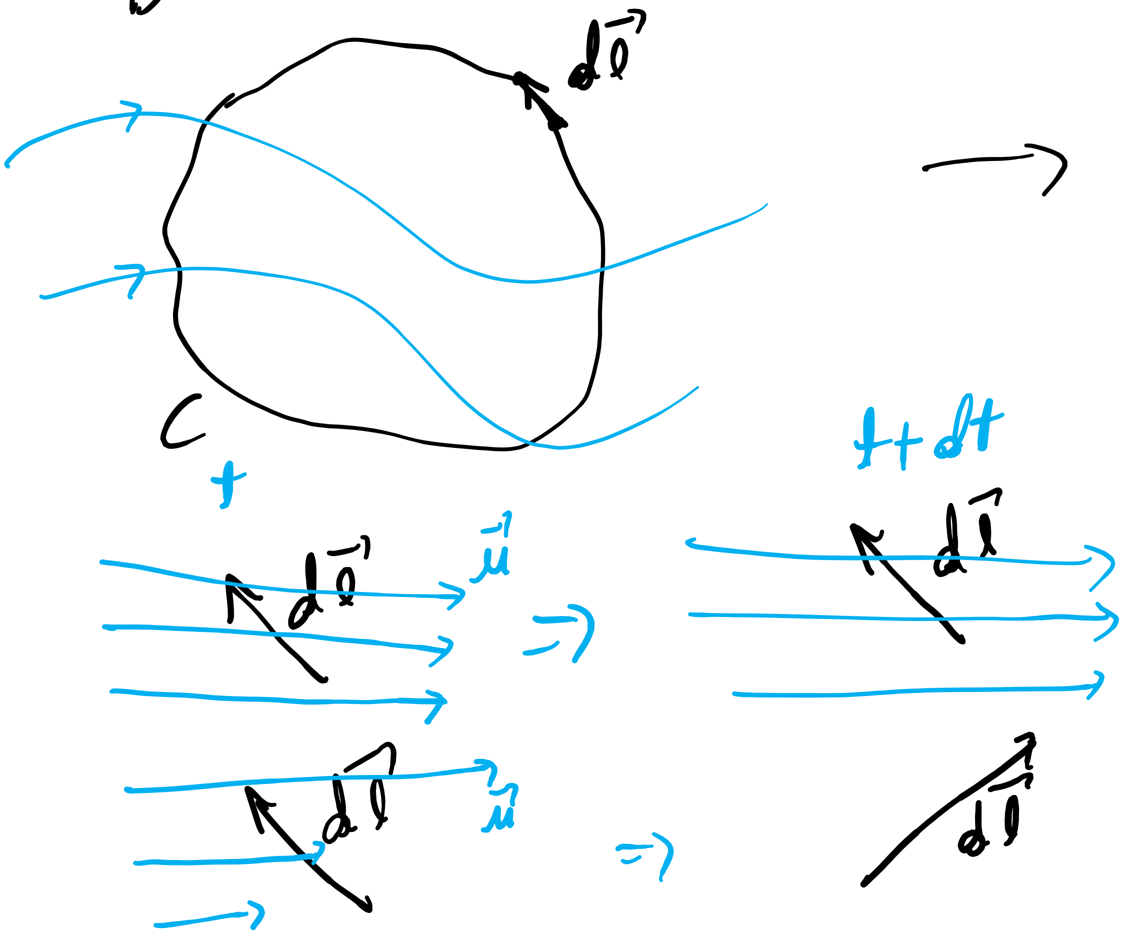
$$= -\int_S \underbrace{\nabla \times \nabla}_{=0}\left(\frac{p}{\rho} + gz\right) \cdot d\vec{S} = 0$$

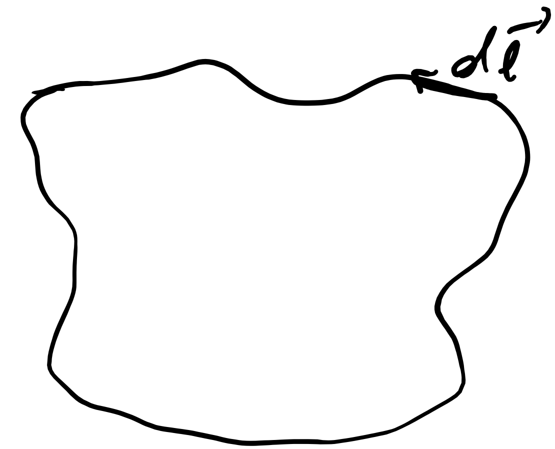
$$I_2 = \oint_c \vec{u} \cdot \underbrace{\frac{D(d\vec{l})}{Dt}}_{\overset{?}{=} d\vec{l} \cdot \nabla\vec{u}} = \oint_c \left[(d\vec{l} \cdot \nabla)\vec{u}\right] \cdot \vec{u}$$

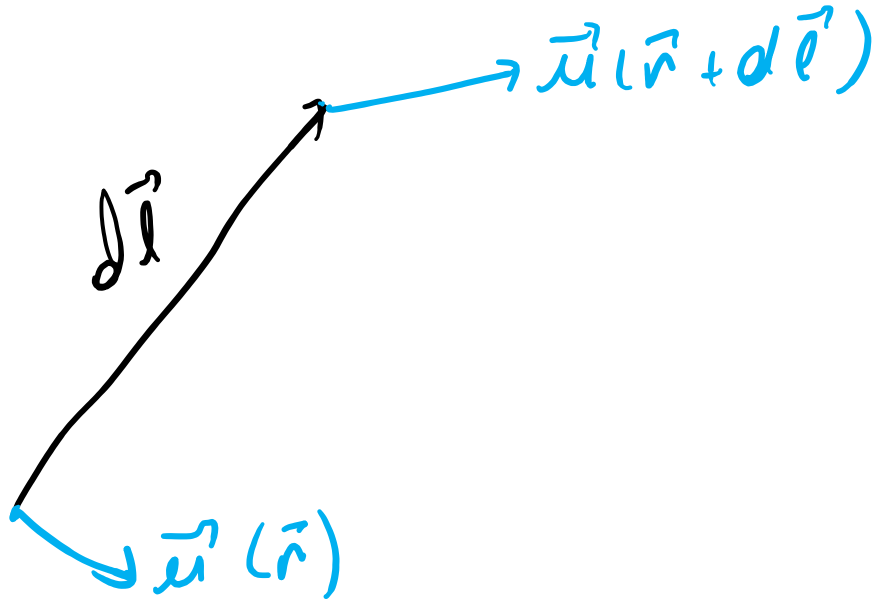
$$= \frac{1}{2}\oint_c d\vec{l} \cdot \nabla(u^2) = \frac{1}{2}\oint_S \nabla \times \nabla(u^2) \cdot d\vec{S} = 0$$

$$\Rightarrow \boxed{\frac{DK}{Dt} = 0}$$

$$\frac{D}{Dt}(d\vec{l}) = d\vec{l} \cdot \nabla \vec{u}$$







$$d(dl_x) = u_x(\vec{r}+d\vec{l})\,dt - u_x(\vec{r})\,dt$$

$$= u_x(x+dl_x,\, y+dl_y)\,dt - u_x(x,y)\,dt$$

$$\approx \cancel{u_x(x,y)\,dt} + \frac{\partial u_x}{\partial x}\,dl_x\,dt + \frac{\partial u_x}{\partial y}\,dl_y\,dt - \cancel{u_x(x,y)\,dt}$$

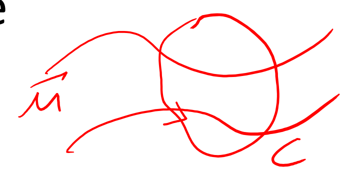
$$\Rightarrow \frac{d}{dt}(dl_x) = d\vec{l}\cdot\nabla u_x \qquad \therefore\ \boxed{\frac{d}{dt}(d\vec{l}) = d\vec{l}\cdot\nabla\vec{u}}$$

Kelvin's circulation theorem

- An ideal fluid that is vorticity free at a given instant is vorticity free at all times.
  $\vec{\omega} = 0$
- Demonstration: see Faber 120-122

- In three dimensions the conservation of vorticity (which corresponds to the conservation of angular momentum in mechanics) takes a somewhat subtle form.

- The circulation of a velocity field is defined to be



$$K\{t\} = \oint \boldsymbol{u}\{\boldsymbol{x}, t\} \cdot \mathrm{d}\boldsymbol{l},$$

where the line is a closed loop which moves with the fluid.

Circulation and vorticity

- By Stokes' theorem

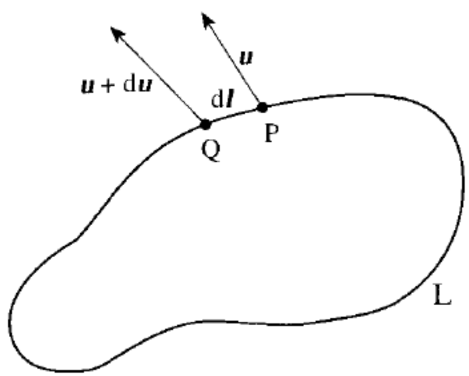
$$K = \oint_{C(t)} \mathbf{u}\cdot d\mathbf{l} = \int_{S(t)} (\underbrace{\boldsymbol{\nabla}\times\mathbf{u}}_{\omega})\cdot \mathbf{n}dS = \int_{S(t)} \Omega\cdot \mathbf{n}dS,$$

where S(t) is a surface whose edges connect with C(t).

K is zero for all loops if Ω is zero in the domain!

Kelvin´s theorem asserts that

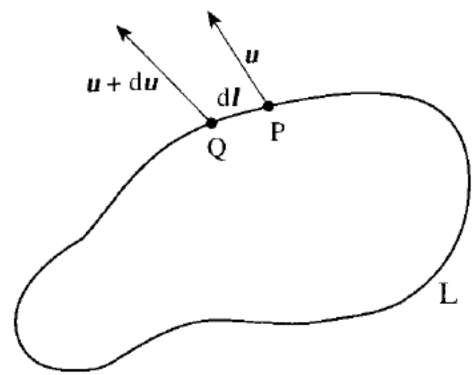
$$\frac{DK}{Dt} = 0.$$



Demonstration

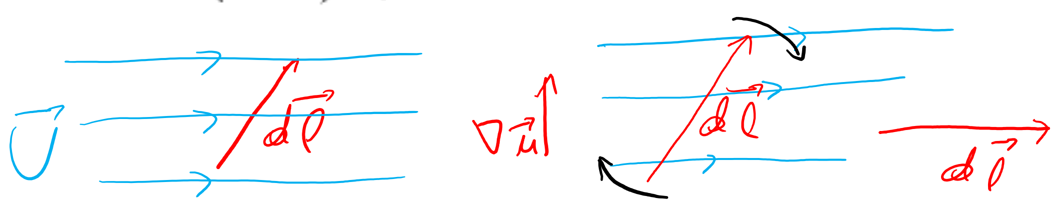
The loop moves with the flow and thus

$$\frac{DK}{Dt} = \oint_C \frac{D\vec{u}}{Dt}\cdot d\vec{l} + \vec{u}\cdot\frac{D(d\vec{l})}{Dt}$$



The second term is the relative velocity of two nearby points on the loop and can be written as $(\partial u/\partial l)dl$.

$$\frac{D(d\vec{l})}{Dt} = d\vec{l}\cdot\nabla\vec{u}$$



Hence

$$\oint_C \vec{u}\cdot\frac{D(d\vec{l})}{Dt} = \oint_C \vec{u}\cdot\left[(d\vec{l}\cdot\nabla)\vec{u}\right] = \frac{1}{2}\oint_C \nabla(\vec{u}^2)\cdot d\vec{l}$$

$$\overset{\text{Stokes}}{=} \frac{1}{2}\int_S (\underbrace{\nabla\times\nabla\vec{u}^2}_{=0})\cdot d\vec{S} = 0$$

If the fluid is incompressible, using Euler:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + gz\right),$$

$$\oint_c \frac{D\vec{u}}{Dt} \cdot d\vec{l} = -\oint_c \nabla\left(\frac{P}{\rho} + gz\right) \cdot d\vec{l}$$

$$\overset{\text{stokes}}{=} -\iint_S \Big[\underbrace{\nabla \times \nabla}_{=0}\left(\frac{P}{\rho} + gz\right)\Big] \cdot d\vec{S} = 0$$

Therefore:

$$\boxed{\frac{\mathrm{D}K}{\mathrm{D}t} = 0}$$

– fluido ideal
($\nu = 0$, $\rho = cte$)

– $\vec{\omega} = 0$

Superposition

- Since the Laplace equation is a linear homogeneous differential equation, the linear combination of two or more solutions of the equation must also be a solution.

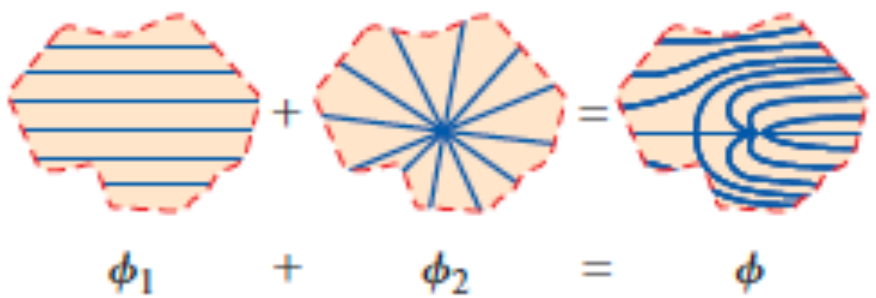
$\phi = A\phi_1 + B\phi_2$

- For example, if ϕ_1 and ϕ_2 are each solutions of the Laplace equation, then $A\,\phi_1 + B\,\phi_2$ are also solutions, where A and B are arbitrary constants.

$\nabla^2\phi = A\nabla^2\phi_1 + B\nabla^2\phi_2$

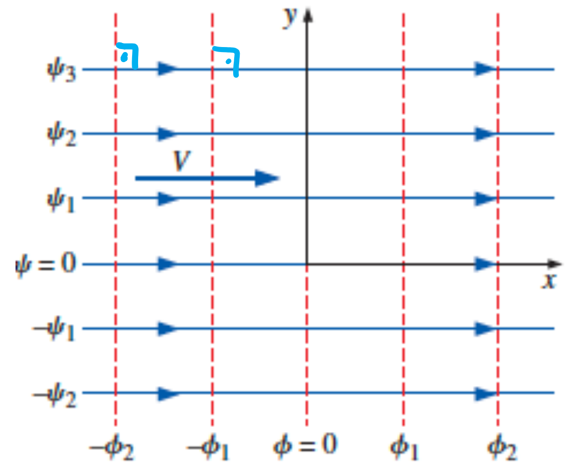
- By extension, you may combine several solutions of the Laplace equation, and the combination is guaranteed to also be a solution.

→ A soluções de $\nabla^2\phi = 0$ são únicas.



Uniform (free) stream

Uniform stream: $u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} = V \qquad v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = 0$

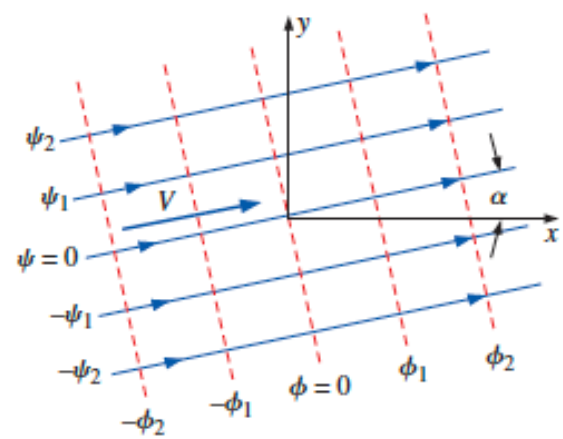


$\phi = Vx + f(y) \quad \rightarrow \quad v = \dfrac{\partial \phi}{\partial y} = f'(y) = 0 \quad \rightarrow \quad f(y) = \text{constant}$

Velocity potential function for a uniform stream: $\phi = Vx$ + Cte

Stream function for a uniform stream: $\psi = Vy$ + Cte

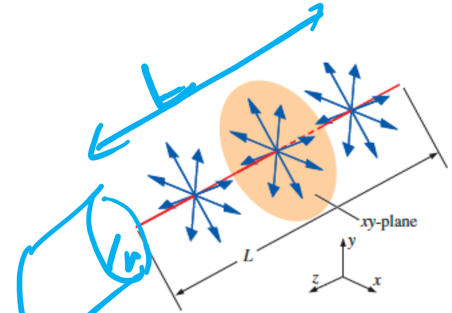
Uniform stream: $\phi = Vr\cos\theta \qquad \psi = Vr\sin\theta$



Uniform stream inclined at angle α:

$$\begin{cases} \phi = V(x\cos\alpha + y\sin\alpha) \\ \psi = V(y\cos\alpha - x\sin\alpha) \end{cases}$$

Line source or sink



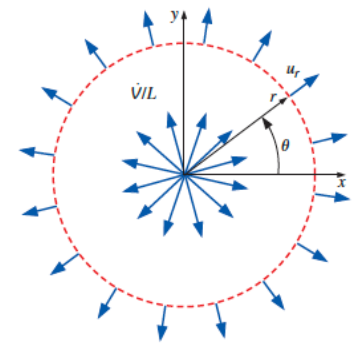
Let the volume flow rate per unit depth, be the line source strength, m

$$\frac{\dot{V}}{L} = 2\pi r u_r \qquad u_r = \frac{\dot{V}/L}{2\pi r}$$

$u_\theta = 0$
$u_z = 0$

$\xi = 2\pi r \cdot L$

The components of the velocity are

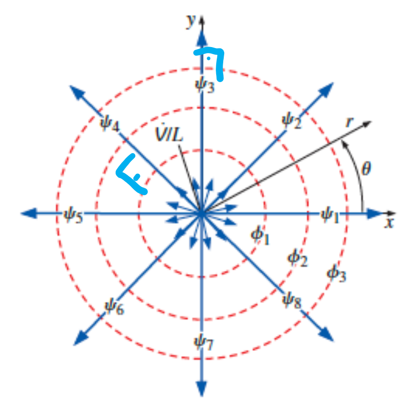


$$\textit{Line source:} \qquad u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{V}/L}{2\pi r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

$$\frac{\partial \psi}{\partial r} = -u_\theta = 0 \quad \rightarrow \quad \psi = f(\theta) \quad \rightarrow \quad \frac{\partial \psi}{\partial \theta} = f'(\theta) = r u_r = \frac{\dot{V}/L}{2\pi}$$

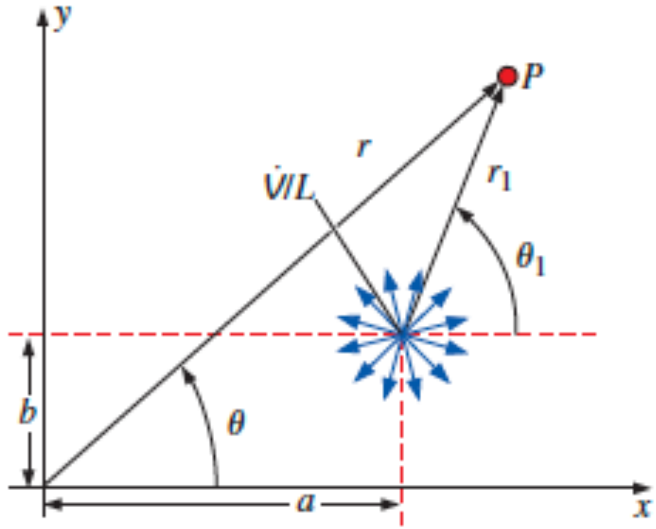
With solution

$$f(\theta) = \frac{\dot{V}/L}{2\pi}\theta + \text{constant}$$



Line source at the origin: $\boxed{\phi = \frac{\dot{V}/L}{2\pi}\ln r}$ and $\boxed{\psi = \frac{\dot{V}/L}{2\pi}\theta}$

Line source or sink at an arbitrary point



Line source at point (a, b):

$$\phi = \frac{\dot{V}/L}{2\pi} \ln r_1 = \frac{\dot{V}/L}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2}$$

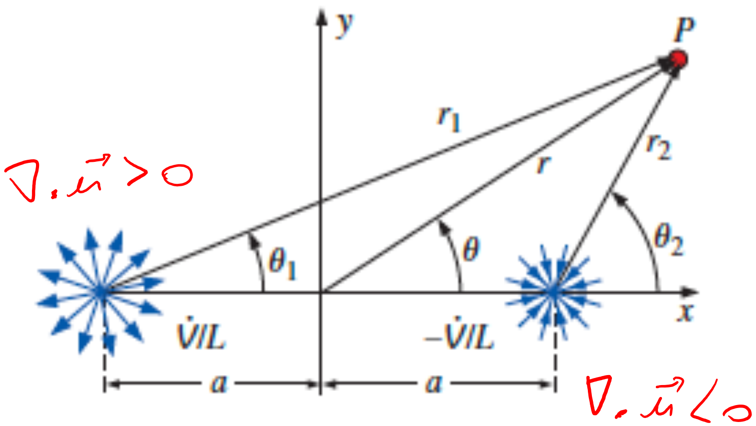
$$\psi = \frac{\dot{V}/L}{2\pi} \theta_1 = \frac{\dot{V}/L}{2\pi} \arctan \frac{y-b}{x-a}$$

Superposition of a source and sink of equal strength

Line source at $(-a, 0)$: $\psi_1 = \frac{\dot{V}/L}{2\pi}\theta_1$ where $\theta_1 = \arctan\frac{y}{x+a}$

Similarly for the sink,

Line sink at $(a, 0)$: $\psi_2 = \frac{-\dot{V}/L}{2\pi}\theta_2$ where $\theta_2 = \arctan\frac{y}{x-a}$



Composite stream function: $\psi = \psi_1 + \psi_2 = \frac{\dot{V}/L}{2\pi}(\theta_1 - \theta_2)$

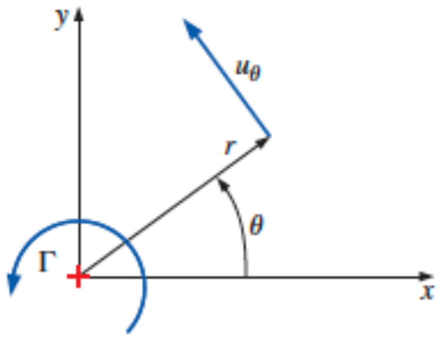
Final result, Cartesian coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ay}{x^2 + y^2 - a^2}$

Final result, cylindrical coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ar\sin\theta}{r^2 - a^2}$

Using

$$\arctan(u) \pm \arctan(v) = \arctan\left(\frac{u \pm v}{1 \mp uv}\right) \pmod{\pi}, \quad uv \neq 1 .$$

Line vortex

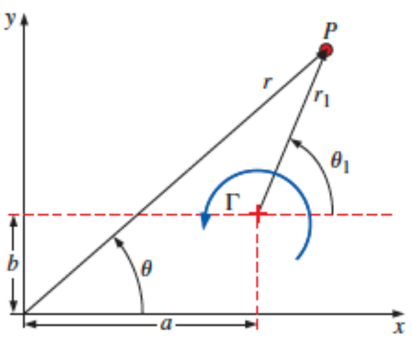


The radial component of the velocity is zero and

Line vortex: $$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

where $\Gamma = 2\pi r u_\theta$, is the circulation, around a loop of radius r.

Then,



Line vortex at the origin: $$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Line vortex at point (a, b):

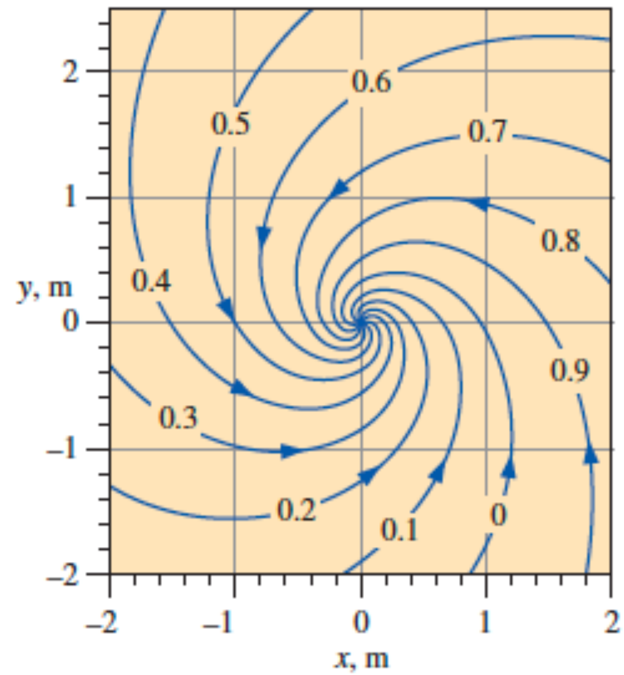
$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\arctan\frac{y-b}{x-a}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2 + (y-b)^2}$$

Superposition of a line sink and a line vortex at the origin

The stream function is

Superposition: $$\psi = \underbrace{\frac{\dot{V}/L}{2\pi}\theta}_{\text{fonte}} - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{vórtice}}$$



with streamlines

Streamlines: $$r = \exp\left(\frac{(\dot{V}/L)\theta - 2\pi\psi}{\Gamma}\right)$$

Velocity components: $$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\dot{V}/L}{2\pi r} \qquad u_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Note that velocity diverges at the origin, which is a singularity (unphysical).

Sources and sinks

(Faber 4.4)

- The 1/R potential $\phi = -\frac{Q}{4\pi R}$ is a solution of Laplace´s equation in 3D
- It describes isotropic flow with velocity $Q/4\pi R^2$
- If Q > 0 it is a source and it is a sink otherwise. Q is the discharge rate.
- Free stream potential $\phi = Ux_1.$
- Superposition of the two gives

$$u_1 = U + \frac{Q}{4\pi R^2}\cos\theta, \quad (u_2^2 + u_3^2)^{1/2} = \frac{Q}{4\pi R^2}\sin\theta,$$

Sources and sinks

- Or in spherical coordinates,

$$u_R = U\cos\theta + \frac{Q}{4\pi R^2}, \quad u_\theta = -U\sin\theta.$$

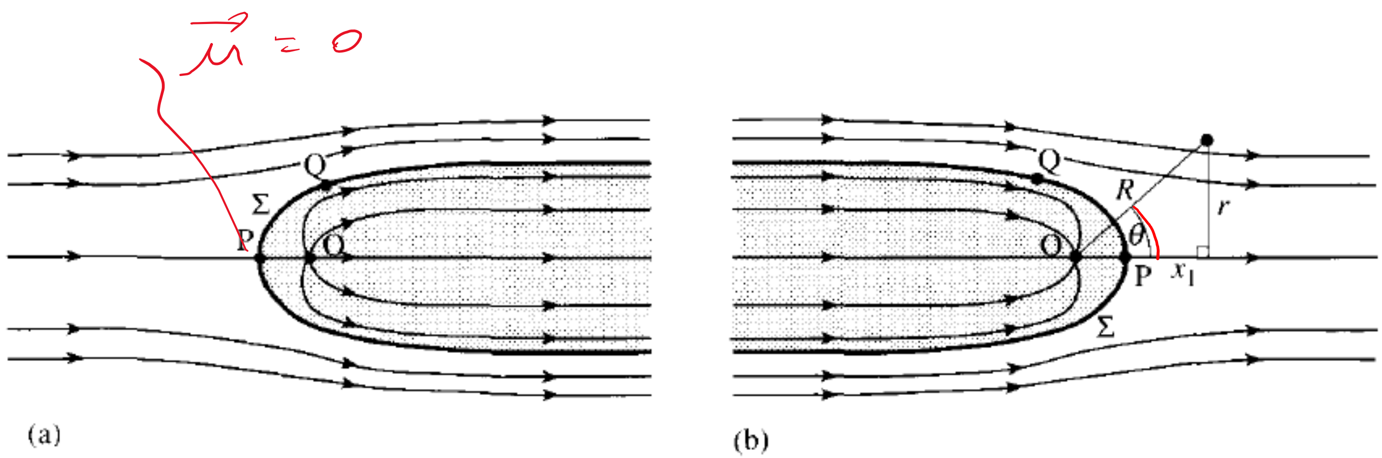
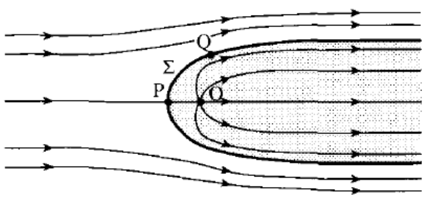


Figure 4.2 Lines of flow past (a) a point source, (b) a point sink. The surface of revolution Σ encloses all the fluid coming from, or destined for, the source or sink respectively.

Excess pressure and force

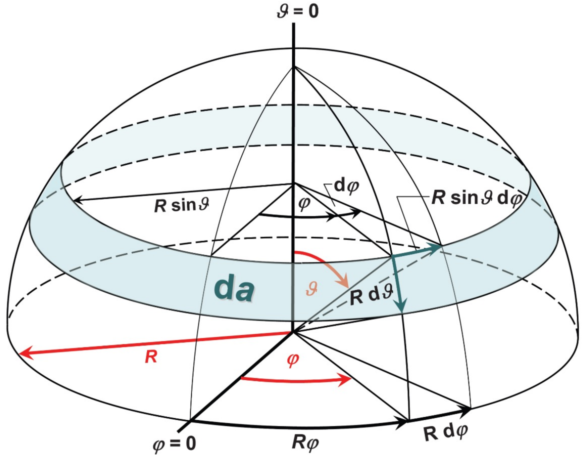
The excess pressure vanishes at infinity where the velocity is that of the free stream. Then Bernoulli gives for the dynamical pressure:



$$p^* = \frac{1}{2}\rho(U^2 - u_R^2 - u_\theta^2) = -\frac{\rho U Q \cos\theta}{4\pi R^2} - \frac{\rho Q^2}{32\pi^2 R^4}$$

Total force in the direction x, exerted by this excess of pressure on the fluid inside a spherical control surface centered on O, of an arbitrary R.

$$\frac{1}{2}\rho UQ \int_0^{\pi} \left(\cos^2\theta \sin\theta + \frac{Q\cos\theta\sin\theta}{8\pi R^2 U}\right) d\theta = \frac{1}{3}\rho UQ$$



Rate of change of momentum

- The total force is equal to the rate of change of momentum in the x direction of the fluid, within the sphere:

$$\sum F_x = \int_0^{\pi} \overbrace{\rho u_1}^{\rho\vec{V}} \underbrace{u_R 2\pi R^2 \sin\theta \, d\theta}_{\vec{V}\cdot\hat{n}\ dA}$$

$$= \int_0^{\pi} \left\{ U^2 \cos\theta + \frac{UQ(1+\cos^2\theta)}{4\pi R^2} + \frac{Q^2 \cos\theta}{16\pi^2 R^4} \right\} 2\pi R^2 \sin\theta \, d\theta$$

$$= \boxed{\frac{4}{3}\rho U Q,}$$

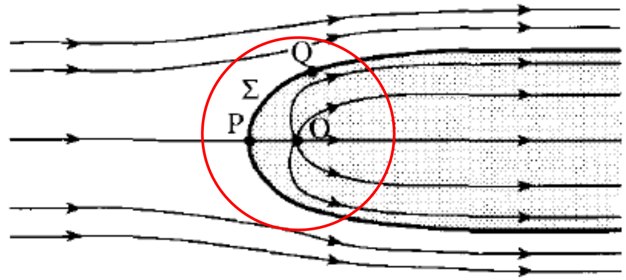
Reynolds transport theorem: $\dfrac{dB_{sys}}{dt} = \dfrac{d}{dt}\displaystyle\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$

$B = b . m$

$$\sum \vec{F} = \frac{d}{dt}\cancel{\int_{CV} \rho \vec{V} \, dV} + \int_{CS} \rho \vec{V}(\vec{V}\cdot\vec{n}) \, dA$$

0

Rate of change of momentum



$p^* \to \frac{1}{3}\rho U Q$

$\sum F_x \to \frac{4}{3}\rho U Q$

- There is then an additional force on the fluid in the x direction of magnitude $\rho U Q$
- This has to be exerted by the source (sink) and thus the source (sink) will experience a reaction force

$$\boxed{\boldsymbol{F} = -\rho U Q.}$$

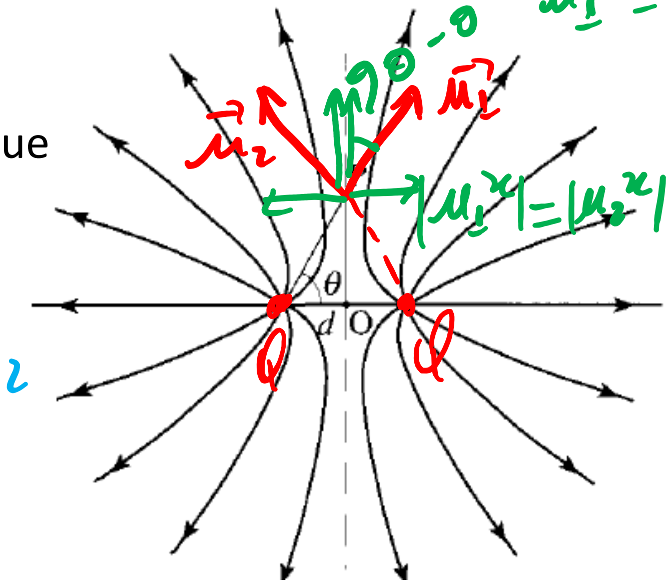
Two equal sources

$u_1^y = |\vec{u}_1| \sin\theta$

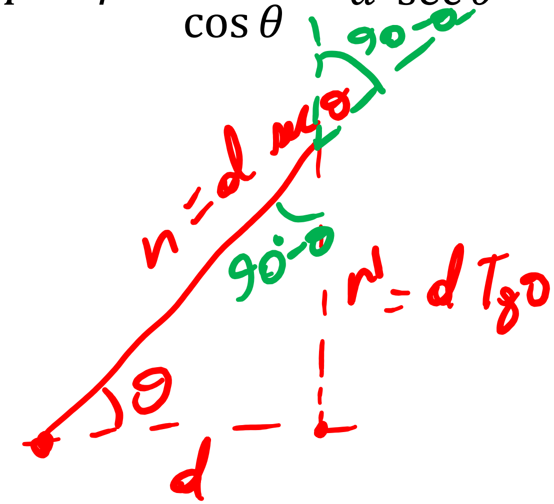
Velocity at one source, due to the other:

$$U = Q/4\pi(2d)^2 = \frac{Q}{16\pi d^2}$$

$$U = \frac{Q}{4\pi r^2}$$



$$OP = r = \frac{d}{\cos\theta} = d\ \sec\theta$$

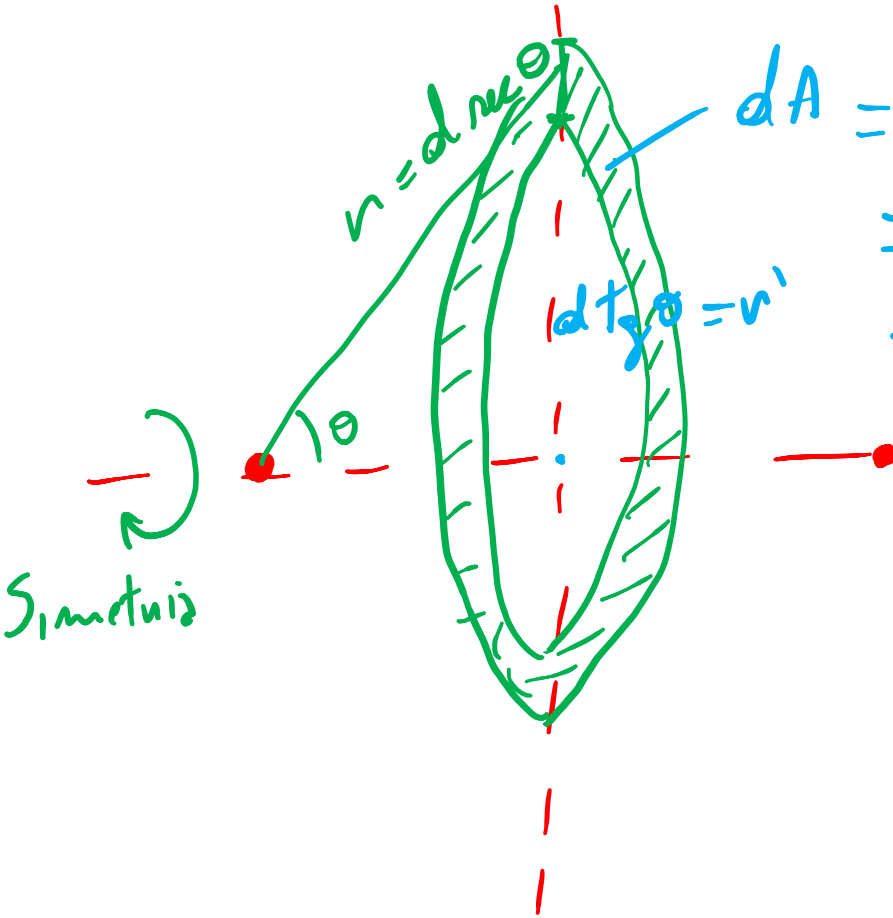


On the plane bissecting the line joining the two sources the normal component of the velocity vanishes. The radial component (in the direction of OP), add and are given by:

$$u_p = 2\sin\theta \cdot \frac{Q}{4\pi(d\sec\theta)^2} = \frac{2Q\sin\theta}{4\pi(d\ \sec\ \theta)^2}.$$

$$Tg\theta = \frac{r'}{d} \Rightarrow r' = d\,tg\theta$$

$$\cos\theta = \frac{d}{r} \Rightarrow r = d\sec\theta$$



$$dA = 2\pi r' dr'$$

$$= 2\pi \, d \, tg\theta \cdot d'(d \, tg\theta)$$

$$= 2\pi \, d \, tg\theta \cdot d \cdot sec^2\theta \, d\theta$$

$$= 2\pi \, d^2 \frac{sen\,\theta}{cos\,\theta} \cdot \frac{1}{cos^2\theta} \, d\theta$$

$$= 2\pi \, d^2 \frac{sen\,\theta}{cos^3\theta} \, d\theta$$

Excess pressure and force

$$H_\infty = H_p \Rightarrow p^* = \frac{\rho}{2}\left(\cancel{U_\infty^2} - u^2\right) = -\frac{\rho}{2}u^2$$

- Assuming that the excess pressure vanishes at infinity, where **u** also vanishes, the excess pressure at P is (Bernoulli),

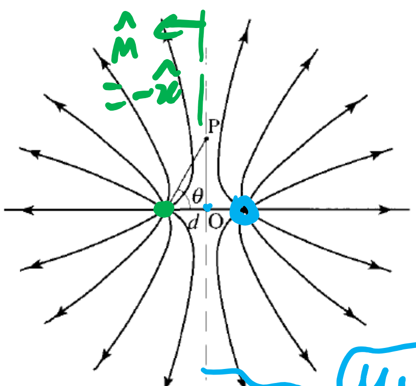
$$p^*\{\theta\} = -\frac{\rho Q^2 \sin^2\theta \cos^4\theta}{8\pi^2 d^4}.$$

- The fluid to the left of the bissecting plane experiences a force due to this excess pressure, given by

$$F_x = -\int_0^\infty p^*\{\theta\}\overbrace{2\pi d \tan\theta\, \mathrm{d}(d\tan\theta)}^{dA} = \frac{\rho Q^2}{4\pi d^2}\int_0^{\pi/2} \sin^3\theta \cos\theta\, \mathrm{d}\theta = \frac{\rho Q^2}{16\pi d^2}.$$

$$= \rho U Q.$$

$=U$



$\hat{n} = -\hat{x}$

$dF_i = \sigma_{ij} n_j dA$, Euler $\sigma_{ij} = -P\delta_{ij}$

$dF_x = -P n_x dA$

$u_\perp = 0$

2 problemas com os mesmos C.C.

$u_\perp = 0$

Analytical solutions of Laplace's equation

(Sec. 4.6)

(i) *Two-dimensional circular polar coordinates (r, θ)*

In this system Laplace's equation becomes

$\nabla^2\phi = 0$

$$\rightarrow \quad r\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\} + \frac{\partial^2\phi}{\partial\theta^2} = 0.$$

$$r^2 \times \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Single-valued solutions in which the variables are separated can readily be found. They are:

$$\phi = \text{constant},$$

$$\phi \propto \phi_0 = \ln r, \tag{4.22}$$

$$\phi \propto \phi_n = \underline{r^n \cos(n\theta)}, \quad or \quad \phi \propto \psi_n = \underline{r^n \sin(n\theta)} \tag{4.23}$$

$$[n = \pm 1, \pm 2, \pm 3 \text{ etc.}].$$

Condições de contorno

Geral:

$$\rightarrow \quad \phi = \text{constant} + A_0\phi_0 + \sum_{n=1}^{\infty}(A_n\phi_n + B_n\psi_n)$$

Ex.:

$$\phi_n = r^n \cos(n\theta)$$

$$\underbrace{r\frac{\partial}{\partial r}\left\{r\frac{\partial\phi}{\partial r}\right\}}_{1^o} + \underbrace{\frac{\partial^2\phi}{\partial\theta^2}}_{2^o} = 0$$

1° termo: $\frac{\partial\phi}{\partial r} = n\, r^{n-1} \cos n\theta$

$$r\frac{\partial}{\partial r}\left[r^n n \cos(\theta n)\right] = r\, n \cos n\theta \cdot n\, r^{n-1}$$

$$= n^2 \cos n\theta \cdot r^n$$

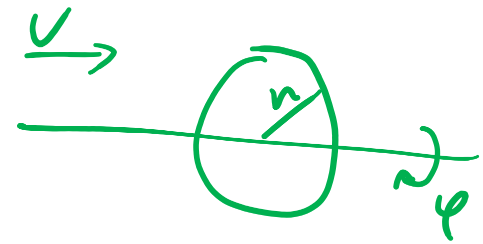
2° termo: $\frac{\partial\phi}{\partial\theta} = -r^n \cdot n \sin(\theta n)$

$$\frac{\partial^2\phi}{\partial\theta^2} = -n^2 r^n \cos(n\theta)$$

1° t. + 2° t. = 0

(iv) *Three-dimensional spherical polar coordinates* (R, θ, ϕ)

Laplace's equation in spherical polars has separated solutions which form a complete set, like the two-dimensional solutions described by (4.22) and (4.23). We need not list them fully here, because we shall be concerned only with problems in which the flow is axially symmetric, i.e. in which the flow potential does not vary with the azimuthal angle ϕ.[2] In these circumstances Laplace's equation simplifies to

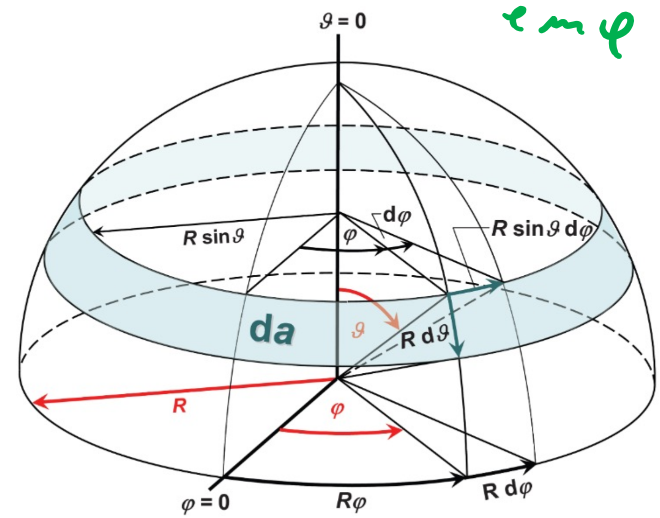


$$\rightarrow \quad \frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right) + \frac{1}{\sin\,\theta}\frac{\partial}{\partial \theta}\left(\sin\,\theta\,\frac{\partial \phi}{\partial \theta}\right) = 0,$$

Simetuia em φ

and its separated solutions may be written as

$$\phi \propto \phi_n^+ = R^n\,\mathrm{P}_n\{\cos\,\theta\},$$
$$\phi \propto \phi_n^- = R^{-(n+1)}\,\mathrm{P}_n\{\cos\,\theta\},$$
$$[n = 0,\ +1,\ +2,\ +3\ \text{etc.}].$$



$$\phi = \sum_m \phi_m^+ \cdot A_m^+ + \phi_m^- B_m^-$$

Laplacian in spherical coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}} \rightarrow 0 \qquad \times r^2$$

The Legendre functions $P_n\{\cos\theta\}$ may be expanded as polynomials in their argument, and we shall need the following expressions in particular:

$$P_0\{\cos\theta\} = 1, \tag{4.29}$$

$$P_1\{\cos\theta\} = \cos\theta, \tag{4.30}$$

$$P_2\{\cos\theta\} = \frac{1}{2}(3\cos^2\theta - 1). \tag{4.31}$$

The full functions ϕ_n^+ and ϕ_n^- are properly called *zonal solid harmonics*. They are orthogonal to one another, and all other solutions of Laplace's equation in three dimensions which share their symmetry (or asymmetry) may be expressed as linear combinations of them [cf. (4.24)].

Some of the solutions described by (4.27) and (4.28) are of course trivial. Thus $\phi_0^+ = 1$ for all values of R and θ. As for

$$\phi_1^+ = \underbrace{R\cos\theta}_{x} = x_1$$

and

$$\phi_0^- = R^{-1},$$

Esc. uniforme

ϕ_1^+

$\vec{u} = U\hat{x} \Rightarrow \phi = U x$

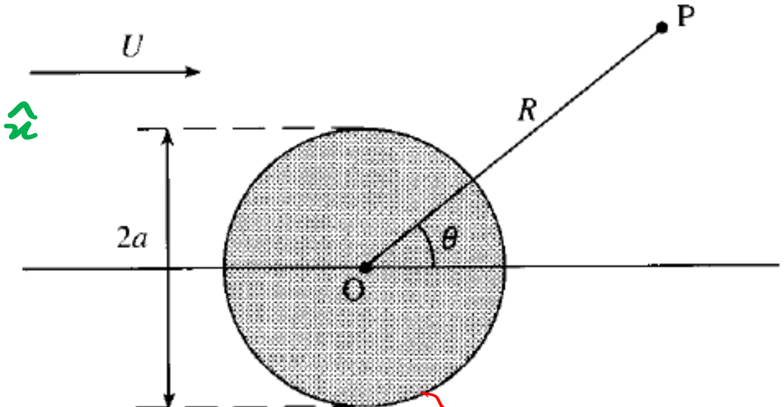
Fonte em 3D

$\vec{u} = \frac{q}{4\pi r^2}\hat{r} \Rightarrow \phi = -\frac{q}{4\pi r} \sim \frac{1}{r}$

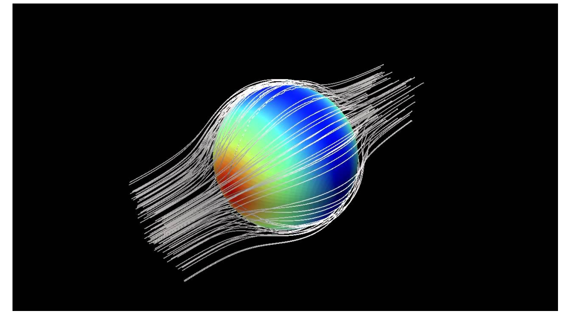
Potential flow around a sphere

Faber 4.7

$\vec{u}_\infty = U\hat{x}$



$u_r = 0$



$$\nabla^2\phi = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

Solutions of the Laplace equation

$$\phi_m^+ = r^m P_m(\cos\theta)$$

$$\phi_m^- = r^{-(m+1)} P_m(\cos\theta)$$

where $P_0 = 1$

$P_1 = \cos\theta$

General solution

$$\phi = \sum_m \left(A_m^+ \phi_m^+ + A_m^- \phi_m^- \right)$$

Boundary condition

1) $r \to \infty \Rightarrow \phi = \underbrace{U}_{A_1^+} \underbrace{r\cos\theta}_{\phi_1^+}$

$\vec{u} = U\hat{x}$

$$\begin{cases} A_1^+ = U \\ A_{m+1}^+ = 0 \end{cases}$$

2) $r = a$, $u_r = 0$ $\Rightarrow$ $\underline{\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = 0}$

$$\frac{\partial \phi_n^-}{\partial r} = -(n+1)\, r^{-(n+2)} \underbrace{P_n(\cos\theta)}_{\cos\theta}$$

Since the solution for a given set of boundary conditions is unique, only $\underline{n=1}$ is needed.

$$\phi_1^- = A_1^- \, r^{-2} \cos\theta$$

Thus

$$\phi = \overbrace{U r \cos\theta}^{\sum A_n^+ \phi_n^+} + \frac{A_1^-}{r^2}\cos\theta = \cos\theta\left(U r + \frac{A_1^-}{r^2}\right)$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = 0 \Rightarrow \boxed{A_1^- = \frac{U a^3}{2}}$$

$$\Rightarrow \quad \phi = \cos\theta \left[Ur + \frac{Ua^3}{2r^2} \right]$$

Velocity

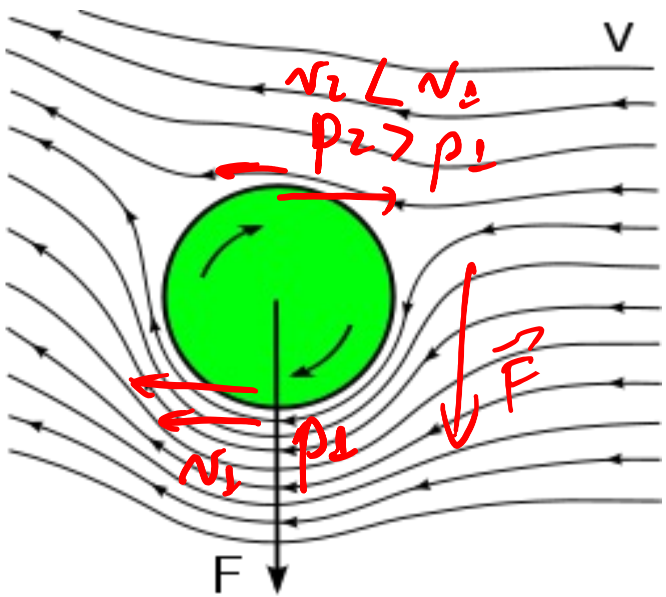
$$u_r = \frac{\partial \phi}{\partial r} = \cos\theta \left[U - \frac{Ua^3}{r^3} \right]$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\operatorname{sen}\theta \left[U + \frac{Ua^3}{2r^3} \right]$$

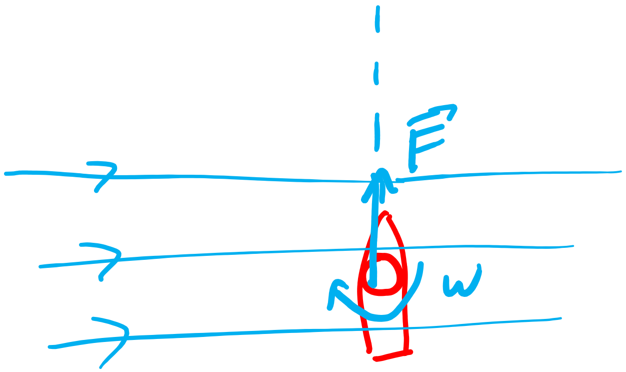
Potential flow around a sphere and Magnus effect

TP $v \uparrow \; p \downarrow$

$$H = \frac{p}{\rho} + \frac{v^2}{2} + g = cte$$



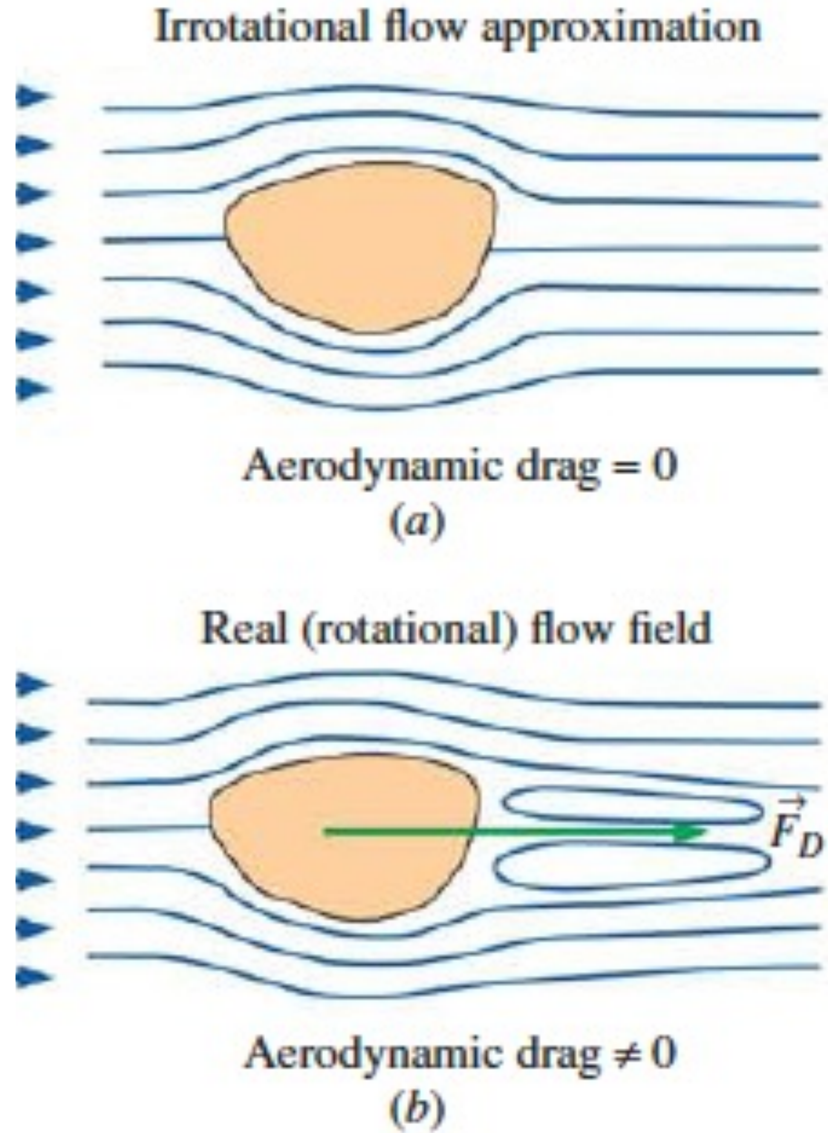
Rotor ship: propulsion by Magnus effect



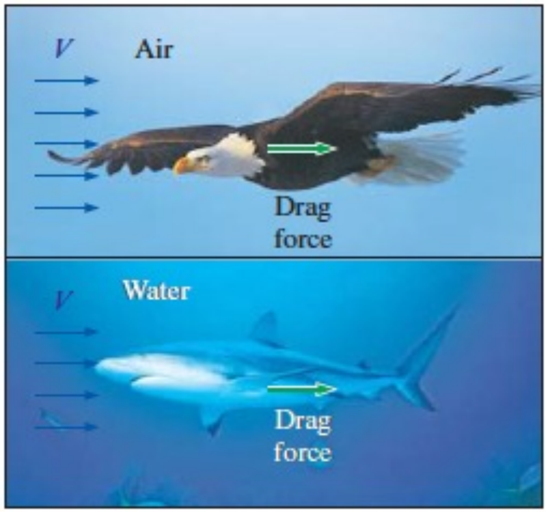
https://en.m.wikipedia.org/wiki/Rotor_ship

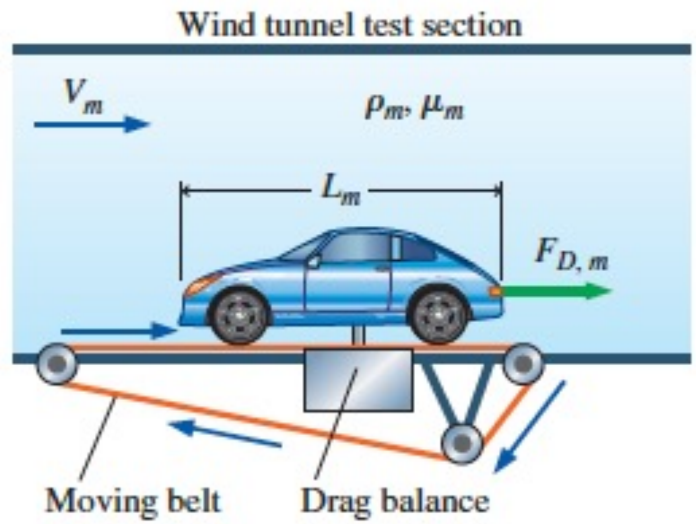
D’Alembert’s paradox: In irrotational flow , the aerodynamic drag force on any body of any shape immersed
in a uniform stream is zero.

“It seems to me that the theory (potential flow), developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers [i.e. mathematicians - the two terms were used interchangeably at that time] to elucidate”



Drag force

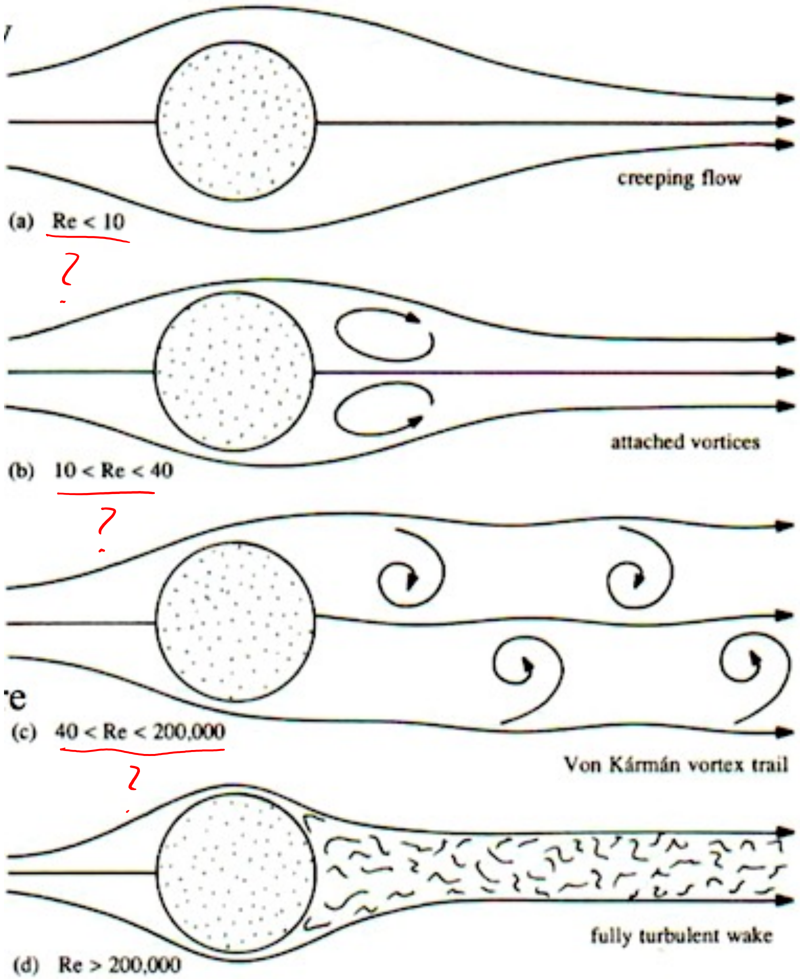


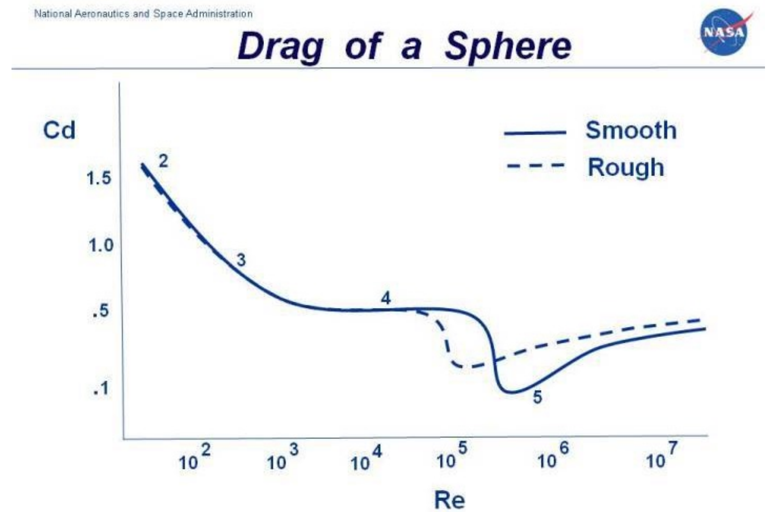


In a real flow, the pressure on the back surface of the body is significantly less than that on the front surface, leading to a nonzero pressure drag on the body. In addition, the no-slip condition on the body surface leads to a nonzero viscous drag as well.

Thus, the irrotational flow falls short in its prediction of aerodynamic drag for two reasons: it predicts no pressure drag and it predicts no viscous drag.

Different regimes





www.youtube.com/watch?v=fcjaxC-e8oY

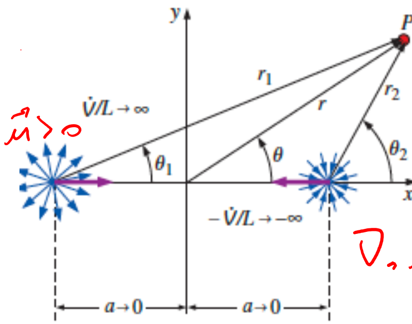
Science of Golf: Why Golf Balls Have Dimples

Doublet: line source and sink close to origin

We have seen before that

Composite stream function: $$\psi = \frac{-\dot{V}/L}{2\pi}\arctan\underbrace{\left[\frac{2ar\sin\theta}{r^2-a^2}\right]}_{\approx \frac{2ar\sin\theta}{r^2-a^2}}$$

$a \to 0$



$\nabla \cdot \vec{u} > 0$

$\nabla \cdot \vec{u} < 0$

By Taylor expanding the arctan around zero:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$Tg^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$a \ll r$

Stream function as $a \to 0$: $$\boxed{\psi \to \frac{-a(\dot{V}/L)r\sin\theta}{\pi(r^2-a^2)}}$$

Doublet: line source and sink close to origin

Let a tend to zero at constant doublet strength K, to find

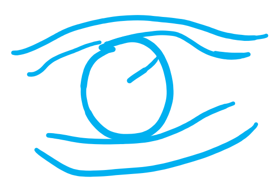
Doublet along the x-axis: $\psi = \dfrac{-a(\dot{V}/L)}{\pi}\dfrac{\sin\theta}{r} = \boxed{-K\dfrac{\sin\theta}{r}}$

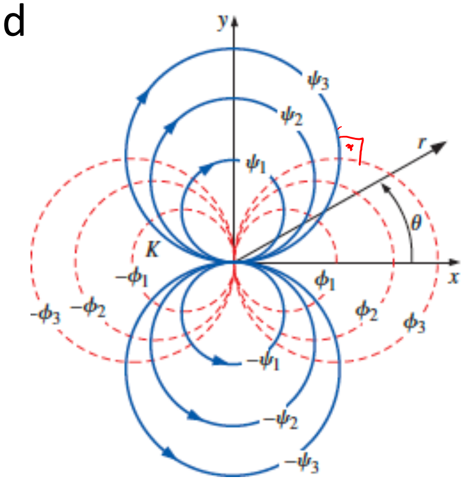
Doublet along the x-axis: $\boxed{\phi = K\dfrac{\cos\theta}{r}}$

$+\ \phi = U r \cos$

(esc. uniforme)

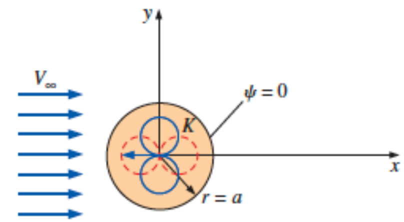
$= \xrightarrow{U}$





Streamlines (solid) and equipotential lines (dashed) for a doublet of strength K located at the origin in the xy-plane and aligned with the x-axis.

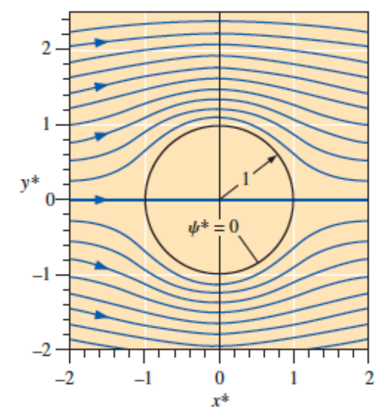
Superposition of a uniform stream and a doublet: Flow over a circular cylinder



Superposition: $$\psi = V_\infty r \sin\theta - K\frac{\sin\theta}{r}$$

For convenience we set $\psi = 0$ when $r = a$

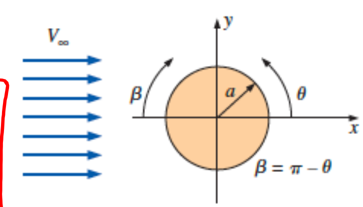
Doublet strength: $$K = V_\infty a^2$$



Alternate form of stream function: $$\boxed{\psi = V_\infty \sin\theta\left(r - \frac{a^2}{r}\right)}$$

$$\psi^* = \sin\theta\left(r^* - \frac{1}{r^*}\right)$$

Nondimensional streamlines: $$r^* = \frac{\psi^* \pm \sqrt{(\psi^*)^2 + 4\sin^2\theta}}{2\sin\theta}$$



$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \boxed{V_\infty\cos\theta\left(1 - \frac{a^2}{r^2}\right)} \qquad u_\theta = -\frac{\partial\psi}{\partial r} = \boxed{-V_\infty\sin\theta\left(1 + \frac{a^2}{r^2}\right)}$$

outra maneira: $\phi = c\ln r + \sum_n A_n \phi_n + B_n \psi_n$